FRACTAL CHARACTERISTICS OF SIMULATED AND LMA-OBSERVED LIGHTNING FLASHES

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Abstract

The fractal dimension of lightning has been studied using both models and observations in the past, but both types of studies suffered from limitations that are now avoidable. Due to a lack of computing power and model sophistication, the fractal dimension of simulated lightning has previously only been studied using either two-dimensional models or unidirectional three-dimensional (3D) models with extremely simplified charge distributions. Also, due to a lack of modern observing networks, previous studies of the fractal dimension of observed lightning were limited to analyzing photographs of flashes in which channels exited the cloud.

This study used the 3D bidirectional lightning model of Mansell et al. (2002), which is derived from the dielectric breakdown models of Niemeyer et al. (1984) and Wiesmann and Zeller (1986). Flashes were simulated in the case of a small, short-lived simulated storm with a realistic charge distribution. The model dynamics were run at 250 m, while the lightning resolution was made as fine as 25 m. The fractal characteristics of these flashes have been analyzed by calculating the correlation dimension using the method originally described in Grassberger and Procaccia (1983). In addition, the fractal characteristics of lightning flashes detected in 3D by the Oklahoma Lightning Mapping Array (OK-LMA) during a small central Oklahoma storm described in Bruning et al. (2007) have also been analyzed using the aforementioned method.

The simulated and observed flashes have been compared and the relationship between correlation dimension and model resolution has been analyzed and used to inform the tuning of parameters in the lightning model.

1. Introduction

It has been known for some time that lightning flashes as well as electrical discharges in general have a structure that can be described using fractal geometry and the concept of fractal dimension (Tsonis and Elsner, 1987; Niemeyer et al., 1984). As noted in Tsonis and Elsner (1987), there is a great deal of variation in the structure of lightning flashes, yet all of the varied structures have certain commonalities that give them the unmistakable appearance of lightning.

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This unmistakable structure common to lightning flashes is due to self-similarity in the branching of the lightning channels at a wide range of scales, and this branching can be quantified using the concept of fractal dimension. Unlike the more common concept of dimension, which is always described by an integer value, fractal dimension can and usually does take on non-integer values. In the case of lightning, the fractal dimension describes how densely branched the flash structure is, and gives a measure of how completely it fills space. For example, if the flash consisted simply of a straight line, its dimension would be equal to 1; on the other hand, if the flash had multiple branches that spread out in a self-similar manner on a plane, it can be described as having a fractal dimension between 1 and 2, with the dimension approaching 2 in the limit of infinitely dense branching (i.e. if the flash completely filled the plane). Similarly, in a three dimensional space, the fractal dimension would be a non-integer value between 1 and 3, with the dimension approaching 3 in the limit of infinitely dense branching.

Numerous studies of the fractal character of lightning and other electrical discharges have been performed in the past, using both numerical models and observations. There were limitations in each of the previous studies that can now be avoided, however. In some of the modeling studies, the charge distributions were much simpler than those present in thunderstorms, and bidirectional discharges like those that occur during lightning were not considered (Niemeyer et al., 1984; Barclay et al., 1990; Sañudo et al., 1995). In another modeling study, realistic thunderstorm charge distributions and bidirectional discharges were simulated, but computational expense limited the researchers to using a two dimensional model (Tan et al., 2006). In the case of the observational study by Tsonis and Elsner (1987), the only data available on the structure of lightning was in the form of photographs, which by their nature only captured two dimensional projections of lightning channels, and, in addition, only captured parts of the flash structure that exited the cloud.

In this study, a fully three-dimensional (3D) bidirectional lightning model was used to simulate lightning flashes within a simulated thunderstorm with a realistic charge distribution. Also, the structure of observed lightning flashes was analyzed using data from a lightning mapping array. In the case of both the modeled and observed flashes, the fractal dimension was calculated using the correlation dimension method.

The model setup, simulated data, and observational data used are described in section 2 of this paper. In section 3 of the paper, the correlation dimension method is described. The results of the correlation dimension calculations and an application of those results are described in section 4. Section 5 concludes the paper.

2. Data

The data used to determine the correlation dimensions in this study came from two sources: 1.) The Oklahoma Lightning Mapping Array (OK-LMA), and 2.) The lightning model of Mansell et al. (2002), used within the NSSL COMMAS model (Mansell et al., 2010).

2.1 *Lightning Model Data*

The model was used to simulate a small storm with a 250 m dynamics grid spacing in both the horizontal and vertical directions. The grid size was 20 km in each horizontal direction and 15 km in the vertical. The initial conditions in the model used a Weisman-Klemp (Weisman and Klemp, 1984) style sounding with a surface mixing ratio of 14.5 g kg^{-1}, a surface temperature of 294 K, and a linear shear profile with 10 m s^{-1} of shear from the surface through 6 km. Finally, convection was initiated using an area of forcing in the lower part of the model domain during the early part of the simulation.

Nine model runs were performed using the same dynamics set up. Within COMMAS, the lightning grid resolution can be set to a fraction of the dynamics grid resolution, and in each of these 9 runs, the lightning grid spacing was made progressively finer, starting at 125 m and continuing on to a spacing of 25 m. The model storms produced a total number of flashes varying from over 200 per run to around 35 per run, with smaller flash rates occurring in the runs with finer grid spacing. Only 20-30 flashes from each run were used in the final analysis for reasons that are discussed below in the Methodology section.

2.2 *OK-LMA Data*

The OK-LMA data used was collected during a small, short-lived central Oklahoma storm (Bruning et al., 2007). The storm produced a mixture of IC and CG lightning flashes, with 30 flashes total. For reasons similar to those that occurred with the model data, only 21 of these flashes were used in the final analysis.

3. Methodology

The fractal dimension was estimated by calculating the correlation dimension (Grassberger and Procaccia, 1983). In essence, the correlation dimension is found by drawing a sphere around each point of the flash, and then determining how the normalized average number of points within the sphere changes as the radius of the sphere is increased. Mathematically, this is done as follows: First, the correlation sum is determined using the correlation integral:

$$C\left(r\right) \equiv \lim_{N\to\infty} \frac{1}{N^2} \sum_{i,j=1}^{N} \theta\left(r - \|\mathbf{X}_i - \mathbf{X}_j\|\right) \qquad (1)$$

In the above equation, $C(r)$ is the correlation sum, N is the total number of points, θ denotes the Heaviside step function, r is the threshold distance (i.e. the radius of the spheres mentioned above). Finally, $\mathbf{X}_i$ and $\mathbf{X}_j$ are the positions of the two points whose interpoint distance is being considered.

If the data set has a fractal character, then the relationship between the correlation sum and the threshold distances should obey a power law,

$$C\left(r\right) \propto r^{\nu} \qquad (2)$$

where the exponent, ν, is called the correlation dimension. Therefore, the correlation dimension is given by the slope of a line fitted to a log-log plot of $C(r)$ vs. r.

When calculating the correlation dimension of empirical fractals, certain issues are inevitable. Various sources of error that can come into play using this method are discussed in Theiler (1990). The two most important sources of error are these: 1.) Any empirical fractal will consist of a finite number of points; therefore, an exact calculation using the infinite limit in equation 1 is not possible. This is mostly an issue because it creates large slope variations in the plot of $C(r)$ vs. r at small threshold distances where the spacing between adjacent points on the fractal is of the same order as the threshold distance. 2.) Any empirical fractal is of a finite size; thus, as the threshold distance approaches the width of the fractal, the sum in the correlation integral approaches N^2, and therefore $C(r)$ approaches 1 and levels off. The effect of both of these issues together is that there exists only a particular threshold distance range, the "scaling range" (Malcai et al., 1997), where the log-log plot of $C(r)$ vs. r is linear. In this study, this was dealt with by using a combination of two methods to automatically determine this range. To eliminate the first issue, points corresponding to increasing values of r on the log-log correlation dimension plot were ignored until the forward-differenced slopes between ten consecutive points varied by less than 1.0. This established a lower bound on the linear scaling range. After this condition was met, the correlation coefficient of the fitted line was recalculated upon the addition of each subsequent point to the plot. When the value of r^2 fell below 0.9998, additional points added to the plot were ignored, thus establishing the upper limit of the linear scaling range. The particular values for the slope variation and correlation coefficient values used were chosen because they gave results in reasonably good agreement with published values when they were used to calculate correlation dimensions for the Henon Map and the Lorenz Attractor.

Upon using this method, some flashes only had linear scaling behavior over a very small range of threshold distances, and thus it would have been difficult to consider the slope of the fitted line to be a reasonable representation of the flash's fractal dimension. Many of the discarded flashes either had a small total number of points or a small flash extent (this was an especially common occurrence in the coarser resolution model runs). Some other discarded flashes had differing fractal characteristics in different parts of the flash. An example of this would be a flash where there are many branches in a bush-like formation in one region of the flash, with one or two isolated channels propagating away from that region. In this case the flash may still have a fractal character, but it is difficult to assign a single fractal dimension value to it. Examples of a both an LMA-detected flash and a modeled flash are shown below in Figure 1, and examples of the correlation dimension plots corresponding to those flashes are shown below in Figure 2.

After the results from the initial set of model runs were found, they were used to set a parameter that affects the flash rate in additional runs, as explained in the discussion below.

4. Results and Discussion

The results of the correlation dimension calculations are shown in Table 1. The average correlation dimension of the model flashes varies with the lightning grid resolution used in the model, with finer resolution runs having higher correlation dimensions and coarser resolution runs having lower correlation dimensions. The standard deviation of the correlation dimensions also varies between the different model resolutions, with the spread generally becoming smaller at finer resolutions. For the LMA-detected flashes, the average correlation dimension falls in the middle of the range of correlation dimensions from the model runs, while the standard deviation is somewhat larger.

Data Set	Average Correlation Dimension	Standard Deviation	Number of Flashes
25.00m model run	1.96	0.04	23
27.78m model run	1.95	0.05	27
31.25m model run	1.95	0.04	22
35.71m model run	1.92	0.07	22
41.67m model run	1.88	0.06	22
50.00m model run	1.85	0.07	24
62.50m model run	1.75	0.06	21
83.33m model run	1.62	0.08	23
125.0m model run	1.39	0.10	28
LMA	1.67	0.14	21

Table 1: Correlation dimension results from Model and LMA data.

The most likely reason for the variations in the fractal dimension of the model runs is as follows: Within the lightning model, the probability of adding adjacent points to a lightning flash is governed by the following equation:

$$p_i\left(E\right) = \begin{cases} \frac{1}{F}\left(E_i - fE_{crit}\right)^{\eta} & for \;\; E_i > fE_{crit} \\ 0 & for \;\; E_i \leq fE_{crit} \end{cases} \tag{3}$$

As explained in Mansell et al. (2002), in the above equation p_i is the probability of adding a particular point to the flash, E_i is the magnitude of the electric field between the ith pair of channel and adjacent nonchannel points, F is a normalization factor found by summing the unnormalized probabilities, and E_{crit} is a critical threshold value for the electric field. Also, f is a factor added to the equation to partially account for the resolution dependent behavior of the flashes, and η is a constant that affects the extent of branching in the flash structure. For all of the model runs in this study, η was set equal to one.

In the model, lighting is modeled as a conducting channel with a diameter equal to that of the grid spacing (Mansell et al., 2002). At coarser resolutions, this leads to the electric field surrounding the conducting channel being lower than it would be at fine resolutions. Thus, E_i will be less likely to exceed fE_{crit} at coarser resolutions. This leads to less branching of the lightning channels as well as smaller overall flash extents, which leads to less energy being dissipated in each flash and thus higher flash rates for a given charging rate. The results of this can be seen in the series plotted in red in Figure 3, where the total number of flashes generated during the simulated storm increases greatly

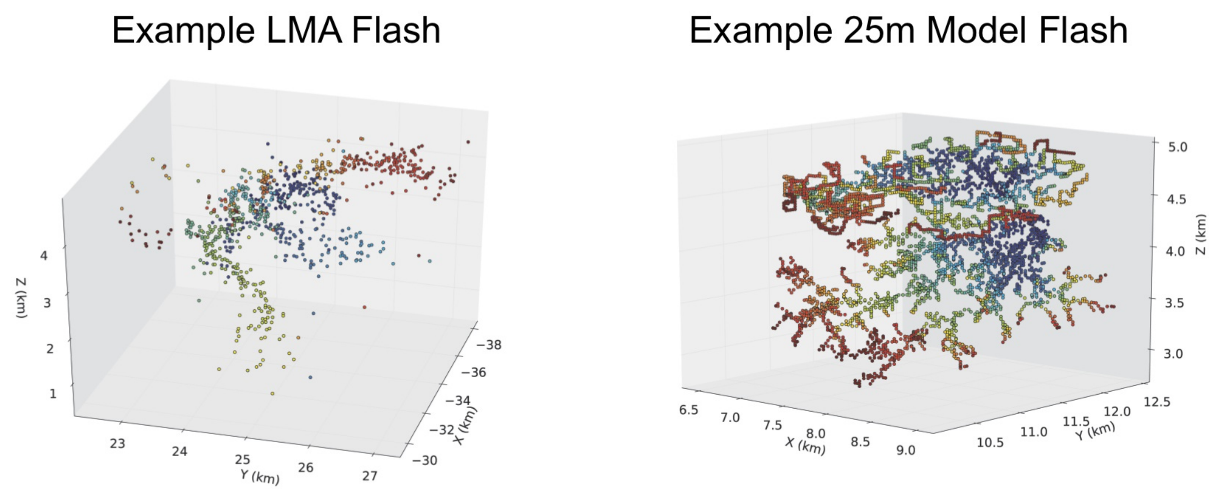

FIG. 1: Spatial plot of source points from an LMA-detected flash and a modeled flash. Blue source points indicate the earliest part of the flash and red source points indicate the latest part of the flash.

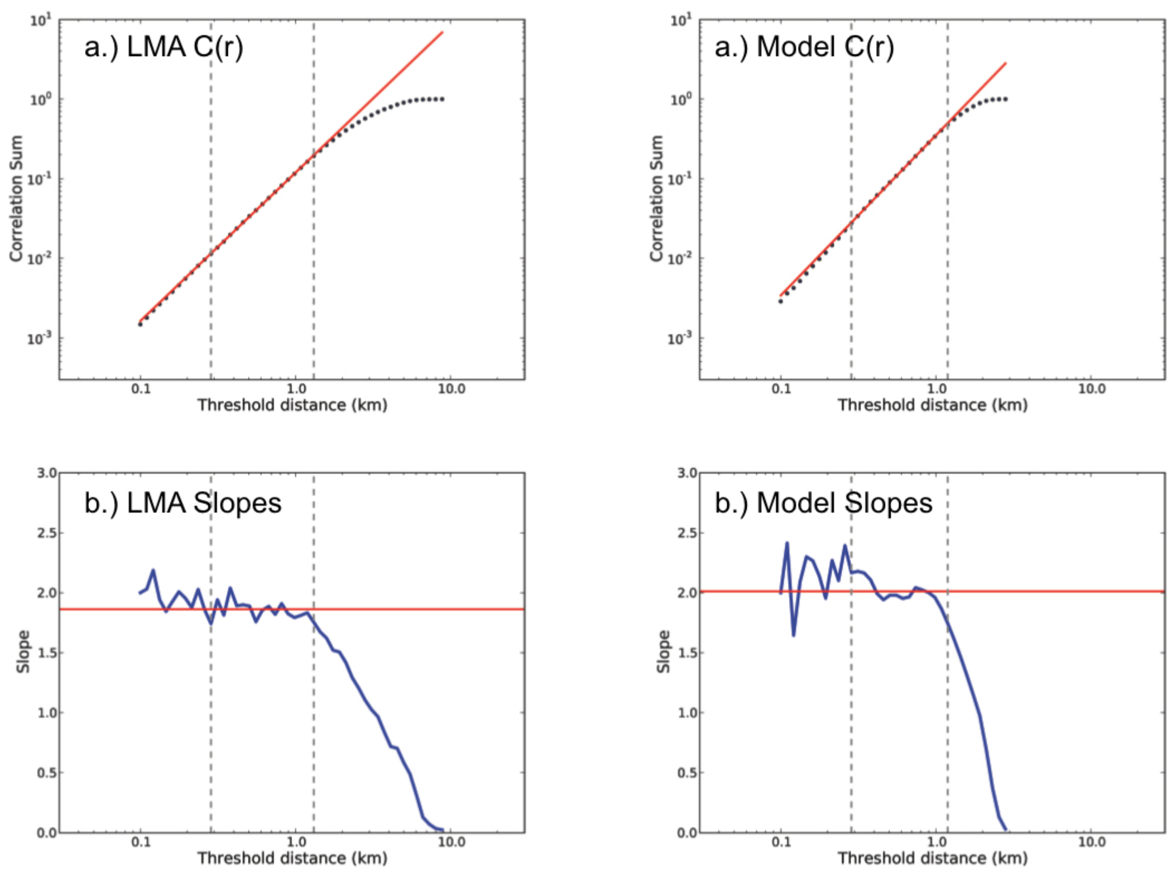

FIG. 2: a.) Log-log plot of C(r) vs. r for the LMA flash (left) and the modeled flash (right). The Solid red line corresponds to a linear least squares fit over the scaling region of each flash. b.) Semi-log plot of forward-differenced slopes between each point vs. r for the LMA flash (left) and modeled flash (right). In all plots the dashed gray lines denote the fractal scaling regions of the flashes where the fits were performed.

as the lightning grid resolution is made coarser. In the model runs used to create the data in that series, the factor f was set to one. Optimally, the total flash number would remain constant at every resolution, and, typically, the value of f would be set based on past experience along with trial and error in an attempt to equalize the flash number. However, in this set of model runs f was always set to one for the sake of comparison between the different resolutions.

Since E_{crit} is multiplied by f, the effect of decreasing the value of f is similar to decreasing the value of E_{crit} in a model that does not include the parameter f. The branching behavior and thus fractal dimension is known to depend on the value of E_{crit} (see references in Mansell et al., 2002). Due to this, it was hypothesized that it may be possible to set the value of f quantitatively using the correlation dimension data. A relationship was found to do this by setting f equal to the ratio of the correlation dimension at each resolution in the set of model runs to the maximum correlation dimension for the set of model runs (i.e. setting $f = \frac{\nu}{max(\nu)}$ and then fitting a line to the points on a plot of correlation dimension ratio vs. model resolution, as shown in Figure 4. Upon doing this, the equation

$$f = \frac{\nu}{max(\nu)} = -0.3003\Delta + 1.0832 \quad (4)$$

was found, where Δ is the lightning grid spacing. This relationship was used to set the value of f in a set of additional model runs, and the blue series in Figure 3 was obtained for total flash number vs. model resolution in the these additional runs. As can be seen, this method of setting f led to a much narrower variation in total flash number across the different resolutions. Additional testing is needed using different storms and different storm types, but this result supports the idea of using equation 4 to set the parameter f.

The correlation dimension results may also be useful for adjusting the value of η. Although the average correlation dimension of the observed flashes falls between that of the 62.5 m and 83.33 m model runs, finer resolution runs are still considered more realistic due to the small diameter of actual lightning channels. Assuming that the linear trend of increasing correlation dimension with finer and finer lightning grid resolution continues, the correlation dimension of a simulated flash with a realistic channel diameter would be greater than the value of 1.96 obtained in the 25 m run. Since this is larger than the average correlation dimension value found for the observed flashes, and since higher values of η lessen the branching of the simulated lightning channel, this could indicate a need to set η to a

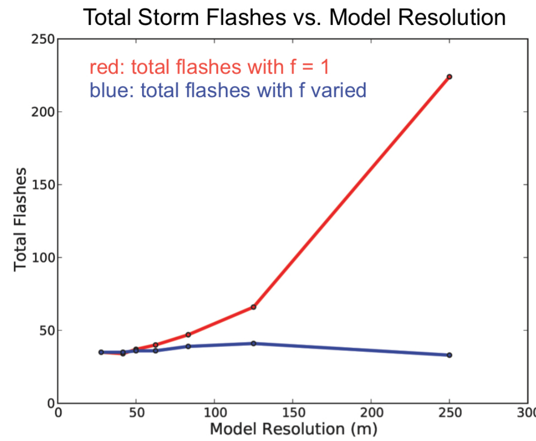

FIG. 3: Total number of flashes throughout the duration of the simulated storm vs. lightning model resolution. The red line uses data from the original set of model runs where f was always set equal to 1, while the blue line uses data from model runs where the value of f was set using equation 4.

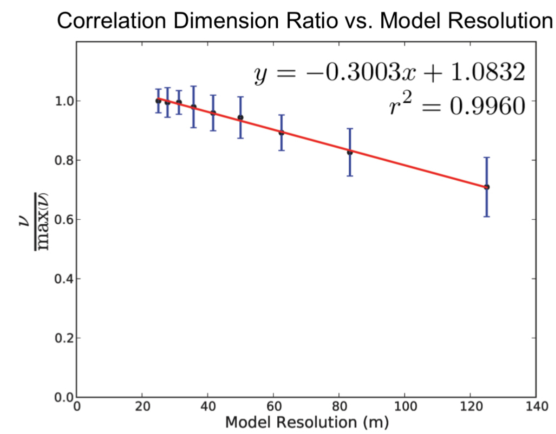

FIG. 4: Ratio of correlation dimension to maximum correlation dimension vs. lightning model resolution for the original set of model runs

value greater than 1.

5. Conclusion

It has been shown in this study that the average correlation dimension of lightning flashes detected by the OK-LMA in a small thunderstorm is 1.67, and that the average correlation dimension of modeled flashes varies depending up the model resolution. An interesting question to explore in future research is whether or not there is any significant variation in the fractal characteristics of lightning between different types of storms, such as supercells and MCSs, or if the correlation dimension value found in this study is typical of all storm types. If there does exist a significant variation, correlation dimension could possibly serve as a discriminator of storm type.

In either case, it is interesting to ask why the correlation dimension of observed flashes has the value that it has. Two-dimensional laboratory electrical discharges in gaseous media have a fractal dimension of 1.7 (Pietronero and Wiesmann, 1984), and 3D discharges should have an even higher fractal dimension. Therefore, the value of 1.67 for 3D flashes is somewhat low based on what might be expected from laboratory discharges. It is possible that this is simply due to poor sampling of the flashes by the lightning mapping array, so it would be interesting to see if the value of 1.67 can be replicated in the future using data from newer lightning mapping arrays with higher temporal resolution than the OK-LMA. If the value can be replicated, one possible reason for its size may lie in the characteristics of the charge structure of thunderstorms. This could provide another pathway for interesting research in the future.

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